IRREDUCIBILITY OF THE GORENSTEIN LOCUS OF THE PUNCTUAL HILBERT SCHEME OF DEGREE 10

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Abstract. Let k be an algebraically closed field of characteristic 0 and let $\mathcal{H}ilb_d^G(\mathbb{P}_k^N)$ be the open locus of the Hilbert scheme $\mathcal{H}ilb_d(\mathbb{P}_k^N)$ corresponding to Gorenstein subschemes. We proved in a previous paper that $\mathcal{H}ilb_d^G(\mathbb{P}_k^N)$ is irreducible for $d \le 9$ and $N \ge 1$. In the present paper we prove that also $\mathcal{H}ilb_{10}^G(\mathbb{P}_k^N)$ is irreducible for each $N \ge 1$, giving also a complete description of its singular locus.

1. Introduction and notation

Let k be an algebraically closed field of characteristic 0 and denote by $\mathcal{H}ilb_d(\mathbb{P}_k^N)$ the Hilbert scheme parametrizing closed subschemes in $\mathbb{P}_k^N$ of dimension 0 and degree d.

On one hand it is well–known that such a scheme is always connected (see [Ha1]) and it is actually irreducible when either $d \ge 1$ and $N \le 2$ (see [Fo] where a more general result is proven) or $d \le 7$ and $N \ge 1$ (see [C–E–V–V]).

On the other hand, in [Ia1] the author proved that, if d is large with respect to N, $\mathcal{H}ilb_d(\mathbb{P}_k^N)$ is always reducible. Indeed for every d and N there always exists a generically smooth component of $\mathcal{H}ilb_d(\mathbb{P}_k^N)$ having dimension dN, the general point of which corresponds to a reduced set of d points but, for d large with respect to $N > 2$, there is at least one other component with general point corresponding to an irreducible scheme of degree d supported on a single point. For example in the above quoted paper [C–E–V–V], the authors also prove the existence of exactly two components in $\mathcal{H}ilb_8(\mathbb{P}_k^N)$, $N \ge 4$.

In view of these results it is reasonable to consider the irreducibility of other naturally occurring loci in $\mathcal{H}ilb_d(\mathbb{P}_k^N)$. E.g. one of the loci that has interested us is the set $\mathcal{H}ilb_d^G(\mathbb{P}_k^N)$ of points in $\mathcal{H}ilb_d(\mathbb{P}_k^N)$ representing schemes which are Gorenstein. This is an important locus since it includes reduced schemes.

A first result, part of the folklore, gives the irreducibility and smoothness of $\mathcal{H}ilb_d^G(\mathbb{P}_k^N)$ when $N \le 3$. In [C–N2] (see also [C–N1]) we proved the irreducibility of $\mathcal{H}ilb_d^G(\mathbb{P}_k^N)$ when $d \le 9$ and $N \ge 1$. In [I–E] the authors stated that $\mathcal{H}ilb_{10}^G(\mathbb{P}_k^N)$ is reducible, essentially by producing an irreducible scheme of dimension 0 and degree 10 corresponding to a point

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in the Hilbert scheme having tangent space of too small dimension. Unfortunately their computations where affected by a numerical mistake as R. Buchweitz pointed out. In [I–K], Lemma 6.21, the authors claim the reducibility of $\mathcal{H}ilb_{14}^G(\mathbb{P}_k^6)$, asserting the existence of numerical examples that can be checked using the "Macaulay"algebra program.

The main result of this paper is the following

Main Theorem. *Let k be an algebraically closed field of characteristic 0. Then the scheme $\mathcal{H}ilb_{10}^G(\mathbb{P}_k^N)$ is irreducible for each $N \ge 1$.* $\square$

In order to prove the above theorem we will also make use of the classification results proved in [C–N2] and [Cs]. The proof of the Main Theorem is given in Section 4. It rests on the analysis of several different cases, which we examine separately in Sections 2, 3, 4.

The idea is that each $X \in \mathcal{H}ilb_{10}^G(\mathbb{P}_k^N)$ is the spectrum of an Artinian Gorenstein k–algebra A and the irreducibility of $\mathcal{H}ilb_{10}^G(\mathbb{P}_k^N)$ depends on some properties of A which can be checked on the direct summands of A, which correspond to the irreducible components of the original scheme X. Thus we can restrict our attention to local algebras A with maximal ideal $\mathfrak{M}$, using all the known classification results.

More precisely in Section 2 we list some preliminary results. In particular we recall that the algebras which we are interested in satisfy $\dim_k(\mathfrak{M}/\mathfrak{M}^2) \le 4$. In Section 3 we examine Artinian, Gorenstein local k–algebras of degree $d \le 10$ for which $\dim_k(\mathfrak{M}^2/\mathfrak{M}^3) \le 3$, with the same methods used in [C–N1], [C–N2] and [Cs]. Artinian, Gorenstein local k–algebras of degree $d \le 10$ with $\dim_k(\mathfrak{M}^2/\mathfrak{M}^3) = 4$ cannot be easily treated in this way so, in Section 4, we analyse this remaining case via an indirect approach.

In the last Section 5 we deal with the singular locus of $\mathcal{H}ilb_{10}^G(\mathbb{P}_k^N)$. Again the fact that $X \in \mathcal{H}ilb_{10}^G(\mathbb{P}_k^N)$ is singular in its Hilbert scheme (we briefly say that X is obstructed in this case) can be recovered from the local direct summands of the associated algebra A. As for the irreducibility we are able to give an easy criterion for deciding weather a fixed scheme X is obstructed or not in term of the underlying algebra.

Notation. In what follows k is an algebraically closed field of characteristic 0.

Recall that a Cohen–Macaulay local ring R is one for which $\dim(R) = \mathrm{depth}(R)$. If, in addition, the injective dimension of R is finite then R is called Gorenstein (equivalently, if $\mathrm{Ext}^i_R\left(M, R\right) = 0$ for each R–module M and $i > \dim(R)$). An arbitrary ring R is called Cohen–Macaulay (resp. Gorenstein) if $R_{\mathfrak{M}}$ is Cohen–Macaulay (resp. Gorenstein) for every maximal ideal $\mathfrak{M} \subseteq R$.

All the schemes X are separated and of finite type over k. A scheme X is Cohen–Macaulay (resp. Gorenstein) if for each point $x \in X$ the ring $\mathcal{O}_{X,x}$ is Cohen–Macaulay (resp. Gorenstein). The scheme X is Gorenstein if and only if it is Cohen–Macaulay and its dualizing sheaf ω_X is invertible.

For each numerical polynomial $p(t) \in \mathbb{Q}[t]$, we denote by $\mathcal{H}ilb_{p(t)}(\mathbb{P}_k^N)$ the Hilbert scheme of closed subschemes of $\mathbb{P}_k^N$ with Hilbert polynomial $p(t)$. With abuse of notation we will denote by the same symbol both a point in $\mathcal{H}ilb_{p(t)}(\mathbb{P}_k^N)$ and the corresponding subscheme of $\mathbb{P}_k^N$. In particular we will say that X is obstructed (resp. unobstructed) in $\mathbb{P}_k^N$ if the corresponding point is singular (resp. non–singular) in $\mathcal{H}ilb_{p(t)}(\mathbb{P}_k^N)$.

Moreover we denote by $\mathcal{H}ilb^G_{p(t)}(\mathbb{P}^N_k)$ the locus of points representing Gorenstein schemes. This is an open subset of $\mathcal{H}ilb_{p(t)}(\mathbb{P}^N_k)$, though not necessarily dense.

If $X \subseteq \mathbb{P}^N_k$ we will denote by $\Im_X$ its sheaf of ideals in $\mathcal{O}_{\mathbb{P}^N_k}$ and we define the normal sheaf of X in $\mathbb{P}^N_k$ as $\mathcal{N}_X := (\Im_X/\Im^2_X)\check{} := \mathcal{H}om_X\big(\Im_X/\Im^2_X, \mathcal{O}_X\big)$. If we wish to stress the fixed embedding $X \subseteq \mathbb{P}^N_k$ we will write $\mathcal{N}_{X|\mathbb{P}^N_k}$ insted of $\mathcal{N}_X$. If $X \in \mathcal{H}ilb_{p(t)}(\mathbb{P}^N_k)$, the space $H^0\big(\mathbb{P}^N_k, \mathcal{N}_X\big)$ can be canonically identified with the tangent space to $\mathcal{H}ilb_{p(t)}(\mathbb{P}^N_k)$ at the point X. In particular X is obstructed in $\mathbb{P}^N_k$ if and only if $h^0\big(\mathbb{P}^N_k, \mathcal{N}_X\big)$ is greater than the local dimension of $\mathcal{H}ilb_{p(t)}(\mathbb{P}^N_k)$ at the point X.

If $\gamma := (\gamma_0, \dots, \gamma_n) \in \mathbb{N}^{n+1}$ is a multi–index, then we set $|\gamma| := \sum_{i=0}^n \gamma_i$, $\gamma! := \prod_{i=0}^n \gamma_i!$, $t^\gamma := t_0^{\gamma_0} \dots t_n^{\gamma_n} \in k[t_0, \dots, t_n]$ and we say that $\gamma \ge 0$ if and only if $\gamma_i \ge 0$ for each $i = 0, \dots, n$. If $\delta := (\delta_0, \dots, \delta_n) \in \mathbb{N}^{n+1}$ is another multi–index then we write $\gamma \ge \delta$ if and only if $\gamma - \delta \ge 0$. Finally we set

$$\binom{\gamma}{\delta} := \frac{\gamma!}{\delta!(\gamma-\delta)!}.$$

For all the other notations and results we refer to [Ha2].

2. Reduction to the local case

We begin this section by recalling some general facts about $\mathcal{H}ilb_d(\mathbb{P}^N_k)$. The locus of reduced schemes $\mathcal{R} \subseteq \mathcal{H}ilb_d(\mathbb{P}^N_k)$ is birational to a suitable open subset of the d–th symmetric product of $\mathbb{P}^N_k$, thus it is irreducible of dimension dN (see [Ia1]). We will denote by $\mathcal{H}ilb^{gen}_d(\mathbb{P}^N_k)$ its closure in $\mathcal{H}ilb_d(\mathbb{P}^N_k)$

Notice that $\mathcal{H}ilb^{gen}_d(\mathbb{P}^N_k)$ is necessarily an irreducible component of $\mathcal{H}ilb_d(\mathbb{P}^N_k)$. Indeed, in any case, we can always assume $\mathcal{H}ilb^{gen}_d(\mathbb{P}^N_k) \subseteq \mathcal{H}$ for a suitable irreducible component $\mathcal{H}$ in $\mathcal{H}ilb_d(\mathbb{P}^N_k)$. If the inclusion were proper then there would exist a flat family with special point in $\mathcal{R}$, hence reduced, and non–reduced general point, which is absurd. We conclude that $\mathcal{H}ilb^{gen}_d(\mathbb{P}^N_k) = \mathcal{H}$.

Definition 2.1. A scheme X is said to be smoothable in $\mathbb{P}^N_k$ if $X \in \mathcal{H}ilb^{gen}_d(\mathbb{P}^N_k)$.

Thus X is smoothable if and only if there exists an irreducible scheme B and a flat family $\mathcal{X} \subseteq \mathbb{P}^N_k \times B \to B$ with special fibre X and general fibre in $\mathcal{R}$, hence reduced. Moreover it is clear that X is smoothable if and only if the same is true for all its connected components (which coincide with its irreducible components since X has dimension 0).

The following result is well–known (see e.g. [C–N2], Lemma 2.2).

Lemma 2.2. *Let X be a scheme of dimension 0 and degree d and let $X \subseteq \mathbb{P}^N_k$ and $X \subseteq \mathbb{P}^{N'}_k$ be two embeddings. Then X is smoothable in $\mathbb{P}^N_k$ if and only if it is smoothable in $\mathbb{P}^{N'}_k$.* $\square$

We now quickly turn our attention to the singular locus of $\mathcal{H}ilb_d(\mathbb{P}^N_k)$. We have (see e.g. [C–N2], Lemma 2.3)

Lemma 2.3. *Let X be a scheme of dimension 0 and degree d and let $X \subseteq \mathbb{P}^N_k$ and $X \subseteq \mathbb{P}^{N'}_k$ be two embeddings. Then*

$$h^0\big(X, \mathcal{N}_{X|\mathbb{P}^N_k}\big) - dN = h^0\big(X, \mathcal{N}_{X|\mathbb{P}^{N'}_k}\big) - dN'. \qquad \square$$

Thanks to the Lemma above it follows that the obstructedness of $X \in \mathcal{H}ilb^{gen}_d(\mathbb{P}^N_k)$ can be checked with respect to an arbitrary embedding $X \subseteq \mathbb{P}^N_k$. Moreover, if $X = \bigcup_{i=1}^p X_i$ where X_i is irreducible of degree d_i, then

$$h^0\big(\mathbb{P}^N_k, \mathcal{N}_X\big) = \sum_{i=1}^p h^0\big(\mathbb{P}^N_k, \mathcal{N}_{X_i}\big), \tag{2.4}$$

thus X is unobstructed if and only if the same is true for all its components X_i.

Now we restrict to $X \in \mathcal{H}ilb^G_d(\mathbb{P}^N_k) \subseteq \mathcal{H}ilb_d(\mathbb{P}^N_k)$ the Gorenstein locus, i.e. the locus of points in $\mathcal{H}ilb_d(\mathbb{P}^N_k)$ representing Gorenstein subschemes of $\mathbb{P}^N_k$. Such a locus is actually open inside $\mathcal{H}ilb_d(\mathbb{P}^N_k)$, since its complement coincides with the locus of points over which the relative dualizing sheaf of the universal family is not invertible. However the locus $\mathcal{H}ilb^G_d(\mathbb{P}^N_k)$ is not necessarily dense.

Trivially $\mathcal{R} \subseteq \mathcal{H}ilb^G_d(\mathbb{P}^N_k)$, i.e. reduced schemes represent points in $\mathcal{H}ilb^G_d(\mathbb{P}^N_k)$. It follows that the main component $\mathcal{H}ilb^{G,gen}_d(\mathbb{P}^N_k) := \mathcal{H}ilb^G_d(\mathbb{P}^N_k) \cap \mathcal{H}ilb^{gen}_d(\mathbb{P}^N_k)$ of $\mathcal{H}ilb^G_d(\mathbb{P}^N_k)$ is irreducible of dimension dN and open in $\mathcal{H}ilb^G_d(\mathbb{P}^N_k)$ since $\mathcal{H}ilb^G_d(\mathbb{P}^N_k)$ is open in $\mathcal{H}ilb_d(\mathbb{P}^N_k)$ (see the introduction).

As first step in the description of $\mathcal{H}ilb^G_d(\mathbb{P}^N_k)$ we show that we can restrict our attention to schemes $X \in \mathcal{H}ilb^G_d(\mathbb{P}^N_k)$ having “big” tangent space at some point. More precisely we have the following (see e.g. [C–N2], Proposition 2.5).

Proposition 2.5. *Let $X \in \mathcal{H}ilb^G_d(\mathbb{P}^N_k)$. If the dimension of the tangent space at every point of X is at most three, then $X \in \mathcal{H}ilb^{G,gen}_d(\mathbb{P}^N_k)$ and it is unobstructed.*

In order to study the obstructedness of $X \in \mathcal{H}ilb^G_d(\mathbb{P}^N_k)$ we finally recall that

$$h^0\big(X, \mathcal{N}_X\big) = \deg(X^{(2)}) - \deg(X) \tag{2.6}$$

where $X^{(2)}$ is the first infinitesimal neighborhood of X in $\mathbb{P}^N_k$ (see Proposition 5.5 of [C–N2]).

From now on we turn our attention from d general to $d = 10$, i.e. we consider $\mathcal{H}ilb^G_{10}(\mathbb{P}^N_k)$. In order to prove its irreducibility it thus suffices to prove the equality $\mathcal{H}ilb^G_{10}(\mathbb{P}^N_k) = \mathcal{H}ilb^{G,gen}_{10}(\mathbb{P}^N_k)$, i.e. that each $X \in \mathcal{H}ilb^G_{10}(\mathbb{P}^N_k)$ is smoothable.

Since we proved in [C–N2] that $\mathcal{H}ilb^G_d(\mathbb{P}^N_k)$ is irreducible if $d \le 9$ and smoothability can be checked componentwise, we deduce the following

Proposition 2.7. *Let $X \in \mathcal{H}ilb_d^G(\mathbb{P}_k^N)$. If all the irreducible components of X have degree at most 9, then $X \in \mathcal{H}ilb_d^{G,gen}(\mathbb{P}_k^N)$.* $\square$

In order to complete the proof of the Main Theorem stated in the introduction, thanks to Propositions 2.5 and 2.7 we thus have to restrict our attention to irreducible schemes X of degree $d = 10$ with tangent space of dimension $n \geq 4$

Each such scheme is isomorphic to $\mathrm{spec}(A)$, where A is a suitable local, Artinian, Gorenstein k–algebra of degree $d = 10$ and $\mathrm{emdim}(A) = n \geq 4$. Thus we will first recall some results about such kind of objects.

Let A be a local, Artinian k–algebra of degree d with maximal ideal $\mathfrak{M}$. In general we have a filtration

$$A \supset \mathfrak{M} \supset \mathfrak{M}^2 \supset \cdots \supset \mathfrak{M}^e \supset \mathfrak{M}^{e+1} = 0$$

for some integer $e \geq 1$, so that its associated graded algebra

$$\mathrm{gr}(A) := \bigoplus_{i=0}^{\infty} \mathfrak{M}^i / \mathfrak{M}^{i+1}$$

is a vector space over $k \cong A/\mathfrak{M}$ of finite dimension $d = \dim_k(A) = \dim_k(\mathrm{gr}(A)) = \sum_{i=0}^{e} \dim_k(\mathfrak{M}^i / \mathfrak{M}^{i+1})$. The Hilbert function of A is by definition the function $h_A \colon \mathbb{N} \to \mathbb{N}$ defined by $h_A(i) := \dim_k(\mathfrak{M}^i / \mathfrak{M}^{i+1})$.

We recall the definition of the *maximum socle degree* of a local, Artinian k–algebra.

Definition 2.8. Let A be a local, Artinian k–algebra. If $\mathfrak{M}^e \neq 0$ and $\mathfrak{M}^{e+1} = 0$ we define the maximum socle degree of A as e and denote it by $\mathrm{msdeg}(A)$.

If $e = \mathrm{msdeg}(A)$ and $n_i := \dim_k(\mathfrak{M}^i / \mathfrak{M}^{i+1})$, $0 \leq i \leq e$, then the Hilbert function h_A of A will be often identified with the vector $(n_0, \dots, n_e) \in \mathbb{N}^{e+1}$.

In any case $n_0 = 1$. Recall that the Gorenstein condition is equivalent to saying that the socle $\mathrm{Soc}(A) := 0 \colon \mathfrak{M}$ of A is a vector space over $k \cong A/\mathfrak{M}$ of dimension 1. If $e = \mathrm{msdeg}(A) \geq 1$ trivially $\mathfrak{M}^e \subseteq \mathrm{Soc}(A)$, hence if A is Gorenstein then equality must hold and $n_e = 1$, thus if $\mathrm{emdim}(A) \geq 2$ we deduce that $\mathrm{msdeg}(A) \geq 2$ and $\deg(A) \geq \mathrm{emdim}(A) + 2$.

Taking into account of Section 5F of [Ia4] (see also [Ia2]), the list of all possible shapes of Hilbert functions of local, Artinian, Gorenstein k–algebra A of degree $d = 10$ and $\mathrm{emdim}(A) \geq 4$ is

$$\begin{gathered} (1,4,1,1,1,1,1),\ (1,5,1,1,1,1),\ (1,6,1,1,1),\ (1,7,1,1),\ (1,8,1) \\ (1,4,2,1,1,1),\ (1,4,2,2,1),\ (1,5,2,1,1),\ (1,6,2,1) \\ (1,4,3,1,1),\ (1,5,3,1), \\ (1,4,4,1). \end{gathered} \tag{2.9}$$

As we will see later on all the above sequences actually occur as Hilbert functions of some local, Artinian, Gorenstein k–algebra. They can be divided into four different families according to $\dim_k(\mathfrak{M}^2 / \mathfrak{M}^3)$.

In the next two sections we will examine separately the two cases $\dim_k(\mathfrak{M}^2 / \mathfrak{M}^3) \leq 3$ and $\dim_k(\mathfrak{M}^2 / \mathfrak{M}^3) = 4$, completing the proof of the Main Theorem.

3. The cases $\dim_k(\mathfrak{M}^2 / \mathfrak{M}^3) \le 3$

When $\dim_k(\mathfrak{M}^2 / \mathfrak{M}^3) = 1$ the sequences on the first line of (2.9) completely characterize the algebra (see [Sa]; another proof can be found in [C–N]), since for a local, Artinian k–algebra A of degree $d \ge n+2$, one has $h_A = (1, n, 1, \dots, 1)$ if and only if $A \cong A_{n,d}$ where

$$A_{n,d} := \begin{cases} k[x_1]/(x_1^d) & \text{if } n = 1, \\ k[x_1, \dots, x_n]/(x_ix_j, x_h^2 - x_1^{d-n})_{\substack{1 \le i < j \le n, \\ 2 \le h \le n}} & \text{if } n \ge 2. \end{cases}$$

Moreover we have

Proposition 3.1. *Let $X \cong \operatorname{spec}(A_{n,d}) \subseteq \mathbb{P}^N_k$, $N \ge n$. Then X is smoothable in $\mathbb{P}^N_k$.*

Proof. By induction on d, it suffices to show that $A_{n,d}$ is a flat specialization of the simpler algebra $A_{n,d-1} \oplus A_{0,1}$, for each $d \ge n+2 \ge 4$ and we refer the reader to Remark 2.10 of [C–N2] for the details. $\square$

We now go to examine the case $\dim_k(\mathfrak{M}^2 / \mathfrak{M}^3) = 2$, i.e. we are considering the sequences on the second line of (2.9). If $h_A = (1, n, 2, 1, \dots, 1)$ (hence $\dim_k(\mathfrak{M}^3 / \mathfrak{M}^4) = 1$) it has been already described in [E–V] (see also Section 3 of [C–N2]). In particular $A \cong A^t_{n,2,d} := k[x_1, \dots, x_n]/I_t$, $t = 1, 2$, where

$$I_1 := \begin{cases} (x_1^2x_2 - x_1^3, x_2^2, x_ix_j, x_h^2 - x_1^3)_{\substack{1 \le i < j \le n,\ 3 \le j \\ 3 \le h \le n}} & \text{if } d = n+4, \\ (x_1^2x_2, x_2^2 - x_1^{d-n-2}, x_ix_j, x_h^2 - x_1^{d-n-1})_{\substack{1 \le i < j \le n,\ 3 \le j \\ 3 \le h \le n}} & \text{if } d \ge n+5, \end{cases}$$

$$I_2 := (x_1x_2, x_2^3 - x_1^{d-n-1}, x_ix_j, x_h^2 - x_1^{d-n-1})_{\substack{1 \le i < j \le n,\ 3 \le j \\ 3 \le h \le n}}.$$

Also in this case we have

Proposition 3.2. *Let $X \cong \operatorname{spec}(A^t_{n,2,d}) \subseteq \mathbb{P}^N_k$, $N \ge n$. Then X is smoothable in $\mathbb{P}^N_k$.*

Proof. See Remark 3.4 of [C–N2]. $\square$

If $h_A = (1, 4, 2, 2, 1)$ (hence $\dim_k(\mathfrak{M}^3 / \mathfrak{M}^4) = 2$) the algebra A can be easily described making use of [Cs], Section 4. In this case $A \cong A^t_{4,2,2,10} := k[x_1, \dots, x_n]/I_t$, $t = 1, 2, 3$, where

$$I_1 := (x_1x_2, x_2^4 - x_1^4, x_ix_j, x_j^2 - x_1^4)_{\substack{1 \le i < j \le 4 \\ 3 \le j}},$$

$$I_2 := (x_1^3x_2 - x_1^4, x_2^2, x_ix_j, x_j^2 - x_1^4)_{\substack{1 \le i < j \le 4 \\ 3 \le j}},$$

$$I_3 := (x_1^3x_2 - x_1^4, x_2^2 - x_1^3, x_ix_j, x_j^2 - x_1^4, x_1^5)_{\substack{1 \le i < j \le 4 \\ 3 \le j}}.$$

Also in this case we have

Proposition 3.3. *Let $X \cong \operatorname{spec}(A^t_{4,2,2,10}) \subseteq \mathbb{P}^N_k$, $N \geq 4$. Then X is smoothable in $\mathbb{P}^N_k$.*

Proof. We will give explicit flat families with general fibre in $\mathcal{H}ilb^{G,gen}_{10}(\mathbb{P}^N_k)$ and special fibre isomorphic to $\operatorname{spec}(A^t_{4,2,2,10})$, $t = 1, 2, 3$. To this purpose take

$$J_1 := (x_1x_2, x_2^4 - x_1^4, x_ix_j, x_3^2 - x_1^4, x_4^2 - bx_4 - x_1^4)_{\substack{1\leq i<j\leq 4\\ 3\leq j}},$$

$$J_2 := (x_1^3x_2 - x_1^4, x_2^2, x_ix_j, x_3^2 - x_1^4, x_4^2 - bx_4 - x_1^4)_{\substack{1\leq i<j\leq 4\\ 3\leq j}},$$

$$J_3 := (x_1^3x_2 - x_1^4, x_2^2 - x_1^3, x_ix_j, x_3^2 - x_1^4, x_4^2 - bx_4 - x_1^4, x_1^5)_{\substack{1\leq i<j\leq 4\\ 3\leq j}}.$$

We claim that the family $\mathcal{A}^t := k[b, x_1, x_2, x_3, x_4]/J_t \to \mathbb{A}^1_k$ has special fibre over $b = 0$ isomorphic to $A^t_{4,2,2,10}$ and general fibre isomorphic to $A^t_{3,2,2,9} \oplus A_{0,1}$. In particular the family $\mathcal{A}^t$ is flat and has general fibre in $\mathcal{H}ilb^{G,gen}_{10}(\mathbb{P}^N_k)$ due to Proposition 2.7, thus it turns out that also its special fibre X is in $\mathcal{H}ilb^{G,gen}_{10}(\mathbb{P}^N_k)$.

It thus remains to prove the claim. To this purpose let us examine only the case $t = 1$, the other ones being similar. Let

$$J_0 := (x_1, x_2, x_3, x_4 - b) \cap (x_1x_2, x_2^4 - x_1^4, x_ix_j, x_3^2 - x_1^4, x_4^2, bx_4 + x_1^4)_{\substack{1\leq i<j\leq 4\\ 3\leq j}}.$$

The inclusion $J_1 \subseteq J_0$ is obvious. Conversely let $y \in J_0$. Then

$$y = u_1(x_2^4 - x_1^4) + u_2(x_3^2 - x_1^4) + u_3x_1x_2 + \sum_{\substack{1\leq i<j\leq 4\\ 3\leq j}} u_{i,j}x_ix_j + vx_4^2 + w(bx_4 + x_1^4)$$

where $u_h, u_{i,j}, vx_4^2, w \in k[b, x_1, x_2, x_3, x_4]$, $h = 1, 2, 3$, $1 \leq i < j \leq 4$ and $3 \leq j$, with the obvious condition $vx_4^2 + wbx_4 \in (x_1, x_2, x_3, x_4 - b)$. Since $x_4 \notin (x_1, x_2, x_3, x_4 - b)$ it follows that $vx_4 + wb \in (x_1, x_2, x_3, x_4 - b)$. With a proper change of the coefficients we can actually assume that $v, w \in k[b, x_4]$ whence we finally obtain $w = -v$, i.e.

$$y = u_1(x_2^4 - x_1^4) + u_2(x_3^2 - x_1^4) + u_3x_1x_2 + \sum_{\substack{1\leq i<j\leq 4\\ 3\leq j}} u_{i,j}x_ix_j + v(x_4^2 - bx_4 - x_1^4)$$

i.e. $y \in J_1$. □

Now we consider the case $\dim_k(\mathfrak{M}^2 / \mathfrak{M}^3) = 3$, i.e. we are considering the sequences on the third line of (2.9). This has been already described in Section 4 of [C–N2] when $d = n + 5$, i.e. $h_A = (1, n, 3, 1)$. In particular $A \cong A^{t,\alpha}_{n,3,n+5} := k[x_1, \dots, x_n]/I_{t,\alpha}$, $t = 1, \dots, 6$, where

$$I_{1,\alpha} := (x_1x_2 + x_3^2, x_1x_3, x_2^2 - \alpha x_3^2 + x_1^2, x_ix_j, x_j^2 - x_1^3)_{\substack{1\leq i<j\leq n\\ 4\leq j}},$$

$$I_{2,0} := (x_1^2, x_2^2, x_3^2 + 2x_1x_2, x_ix_j, x_j^2 - x_1x_2x_3)_{\substack{1\leq i<j\leq n\\ 4\leq j}},$$

$$I_{3,0} := (x_1^2, x_2^2, x_3^2, x_ix_j, x_j^2 - x_1x_2x_3)_{\substack{1\leq i<j\leq n\\ 4\leq j}},$$

$$I_{4,0} := (x_2^3 - x_1^3, x_3^3 - x_1^3, x_ix_j, x_j^2 - x_1^3)_{\substack{1\leq i<j\leq n\\ 4\leq j}},$$

$$I_{5,0} := (x_1^2, x_1x_2, x_2x_3, x_2^3 - x_3^3, x_1x_3^2 - x_3^3, x_ix_j, x_j^2 - x_3^3)_{\substack{1\leq i<j\leq n\\ 4\leq j}},$$

$$I_{6,0} := (x_1^2, x_1x_2, 2x_1x_3 + x_2^2, x_3^3, x_2x_3^2, x_ix_j, x_j^2 - x_1x_3^2)_{\substack{1\leq i<j\leq n\\ 4\leq j}}.$$

Also in this case we have

Proposition 3.4. *Let $X \cong \operatorname{spec}(A^{t,\alpha}_{n,3,n+5}) \subseteq \mathbb{P}^N_k$, $N \geq n$. Then X is smoothable in $\mathbb{P}^N_k$.*

Proof. See Remark 4.9 of [C–N2]. $\square$

If $h_A = (1,4,3,1,1)$, the algebra A can be described making use of [Cs], Section 5. In this case $A \cong A^t_{4,3,10} := k[x_1,\dots,x_n]/I_t$, $t = 0,\dots,6$, where

$$\begin{aligned}
I_0 &:= (x_1x_2 + x_3^2, x_1x_3, x_2^2 - x_1^3, x_ix_4, x_4^2 - x_1^4)_{1\leq i\leq 3},\\
I_1 &:= (x_1x_2 + x_3^2, x_1x_3, x_2^2 - x_3^2 - x_1^3, x_ix_4, x_4^2 - x_1^4)_{1\leq i\leq 3},\\
I_2 &:= (x_1x_2, x_1^2 - x_3^3, x_2^2 - x_3^3, x_1x_3^2, x_2x_3^2, x_ix_4, x_4^2 - x_3^4)_{1\leq i\leq 3},\\
I_3 &:= (x_1x_2, x_2x_3, x_1^2 - x_3^3, x_1x_3^2, x_2^3 - x_3^4, x_ix_4, x_4^2 - x_3^4)_{1\leq i\leq 3},\\
I_4 &:= (x_1x_2, x_1x_3, x_2x_3, x_2^3 - x_1^4, x_3^3 - x_1^4, x_ix_4, x_4^2 - x_1^4)_{1\leq i\leq 3},\\
I_5 &:= (x_1x_2, x_2x_3, x_1^2, x_1x_3^2 - x_2^4, x_3^3 - x_2^4, x_ix_4, x_4^2 - x_2^4)_{1\leq i\leq 3},\\
I_6 &:= (x_1x_2 - x_3^3, 2x_1x_3 + x_2^2, x_1^2, x_1x_3^2, x_2x_3^2, x_ix_4, x_4^2 - x_3^4)_{1\leq i\leq 3}.
\end{aligned}$$

Again we have

Proposition 3.5. *Let $X \cong \operatorname{spec}(A^t_{4,3,10}) \subseteq \mathbb{P}^N_k$, $N \geq 4$. Then X is smoothable in $\mathbb{P}^N_k$.*

Proof. The argument is the same of the proof of Proposition 3.2. Indeed it suffices to take

$$\begin{aligned}
J_0 &:= (x_1x_2 + x_3^2, x_1x_3, x_2^2 - x_1^3, x_ix_4, x_4^2 - bx_4 - x_1^4)_{1\leq i\leq 3},\\
J_1 &:= (x_1x_2 + x_3^2, x_1x_3, x_2^2 - x_3^2 - x_1^3, x_ix_4, x_4^2 - bx_4 - x_1^4)_{1\leq i\leq 3},\\
J_2 &:= (x_1x_2, x_1^2 - x_3^3, x_2^2 - x_3^3, x_1x_3^2, x_2x_3^2, x_ix_4, x_4^2 - bx_4 - x_3^4)_{1\leq i\leq 3},\\
J_3 &:= (x_1x_2, x_2x_3, x_1^2 - x_3^3, x_1x_3^2, x_2^3 - x_3^4, x_ix_4, x_4^2 - bx_4 - x_3^4)_{1\leq i\leq 3},\\
J_4 &:= (x_1x_2, x_1x_3, x_2x_3, x_2^3 - x_1^4, x_3^3 - x_1^4, x_ix_4, x_4^2 - bx_4 - x_1^4)_{1\leq i\leq 3},\\
J_5 &:= (x_1x_2, x_2x_3, x_1^2, x_1x_3^2 - x_2^4, x_3^3 - x_2^4, x_ix_4, x_4^2 - bx_4 - x_2^4)_{1\leq i\leq 3},\\
J_6 &:= (x_1x_2 - x_3^3, 2x_1x_3 + x_2^2, x_1^2, x_1x_3^2, x_2x_3^2, x_ix_4, x_4^2 - bx_4 - x_3^4)_{1\leq i\leq 3},
\end{aligned}$$

observing again that $\mathcal{A}^t := k[b,x_1,x_2,x_3,x_4]/J_t \to \mathbb{A}^1_k$ is flat, it has special fibre over $b = 0$ isomorphic to $A^t_{4,3,10}$ and general fibre isomorphic to $A^t_{3,3,9} \oplus A_{0,1}$. $\square$

Remark 3.6. In Section 4 of [Cs], local, Artinian, Gorenstein k–algebras A with Hilbert function $h_A = (1,n,2,\dots,2,1)$ are completely classified. Taking into account of such a classification, it is trivial to modify the above explicit proof of Proposition 3.3 in order to prove that every scheme $X \cong \operatorname{spec}(A)$ with $h_A = (1,n,2,\dots,2,1)$ is smoothable for each $n \geq 2$.

Similarly, it is trivial to modify the proof Proposition 3.5 in order to prove that every scheme $X \cong \operatorname{spec}(A)$ with $h_A = (1,n,3,1,\dots,1)$ is smoothable for each $n \geq 3$.

4. The case $\dim_k(\mathfrak{M}^2\,/\,\mathfrak{M}^3)\,=\,4$

In this section we deal with the last case, namely $\dim_k(\mathfrak{M}^2\,/\,\mathfrak{M}^3)=4$ or, equivalently, $h_A=(1,4,4,1)$. In this case we will not exploit any explicit description for such algebras as we did in the case $\dim_k(\mathfrak{M}^2\,/\,\mathfrak{M}^3)\le 3$ but we will make use of some classical results about Artinian Gorenstein k–algebras combined with a recent structure Theorem for such algebras A with $h_A=(1,N,N,1)$ (see [E–R]).

Indeed on one hand Theorem 3.3 of [E–R] states that each Artinian, Gorenstein k–algebras with $h_A=(1,N,N,1)$ is canonically graded, i.e. there exists an homogeneous ideal $I\subseteq S:=k[x_1,\dots,x_N]$ such that $A\cong S/I$.

On the other hand, in order to construct such graded quotient algebras it suffices to make use of the theory of inverse systems that we are going to recall very quickly (as reference see [I–K], Section 1). We have an action of $S:=k[x_1,\dots,x_N]$ over $R:=k[y_1,\dots,y_N]$ given by partial derivation by identifying x_i with $\partial/\partial y_i$. Hence

$$x^\alpha(y^\beta):=\begin{cases}\alpha!\binom{\beta}{\alpha}y^{\beta-\alpha} & \text{if } \beta\ge\alpha,\\ 0 & \text{if } \beta\not\ge\alpha.\end{cases}$$

Such an action defines a perfect pairing $S_d\times R_d\to k$ between forms of degree d in R and in S. We will say that two *homogeneous forms $g\in R$ and $f\in S$ are apolar if $f(g)=0$*. As explained in [I–K] apolarity allows us to associate an Artinian Gorenstein graded quotient of S to a form in R as follows. Let $g\in R_d$: then we set

$$g^\perp:=\{\ f\in S\mid f(g)=0\ \}$$

and it is easy to prove that both $g^\perp$ is a homogeneous ideal in S and $S/g^\perp$ is an Artinian Gorenstein graded quotient of S with socle in degree d. Also the converse is true i.e. if A is an Artinian Gorenstein graded quotient of S, say $A:=S/I$, with socle in degree d then there exists $g\in R_d$ such that $I=g^\perp$. The main result about apolarity due to Macaulay (see [I–K], Lemma 2.12 and the references cited there) is the following

Theorem 4.1. *The map $g\mapsto S/g^\perp$ induces a bijection between $\mathbb{P}(R_d)$ and the set of graded Artinian Gorenstein quotient rings of S with socle in degree d.* □

Moreover the set of polynomials corresponding to algebras A having maximal embedding dimension $h_A(1)=N$ is a non–empty open subset of $\mathbb{P}(R_d)$ due to the following standard and well–known

Lemma 4.2. *Let $g\in R_d$, $A:=S/g^\perp$, $t\le N$. Then $h_A(1)\le t$ if and only if there exist $\ell_1,\dots,\ell_t\in R_1$ such that $g\in k[\ell_1,\dots,\ell_t]$.*

Proof. If $t=N$ there is nothing to prove. Assume that $t<N$. If $h_A(1)\le t$, up to a proper change of the coordinates $x_1,\dots,x_N\in S_1$ we can assume that $x_N\in g^\perp$, thus $g\in k[y_1,\dots,y_{N-1}]$. Conversely if there exist $\ell_1,\dots,\ell_t\in R_1$ such that $g\in k[\ell_1,\dots,\ell_t]$, since $\dim_k(S_1)=N$, it follows the existence of linear forms $\ell_{t+1},\dots,\ell_N\in S_1$ which are

not in the space spanned by $\ell_1, \dots, \ell_t$: in particular $\ell_i(g) = 0$ for such $N-t$ forms. Thus $\ell_{t+1}, \dots, \ell_N \in g^\perp$, whence $h_A(1) = \dim_k(S_1) - \dim_k(g^\perp \cap S_1) \le t$ $\square$

Now, we restrict our attention to algebras with Hilbert function $(1,4,4,1)$. Thus there exists a natural variety $\mathcal{Z}$ which parametrizes such kind of algebras. More precisely $\mathcal{Z}$ is the open non–empty subset of $\mathbb{P}(R_3) \cong \mathbb{P}^{19}_k$ of cubic surfaces in $\mathbb{P}^3_k$ which are not cones due to the previous lemma.

From now on we will denote by $\mathcal{Z}_N$ the locus of irreducible schemes $X \in \mathcal{H}ilb_{10}^G(\mathbb{P}^N_k)$ of the form $X = \operatorname{spec}(A) \subseteq \mathbb{P}^N_k$ with $h_A = (1,4,4,1)$. Necessarily $N \ge 4$ and $\mathcal{Z}_4 = \mathcal{Z}$ thus it is irreducible.

Our aim is to prove that $\mathcal{Z}_N \subseteq \mathcal{H}ilb_{10}^{G,gen}(\mathbb{P}^N_k)$. Let us examine first the case $N = 4$. If the closure of $\mathcal{Z}_4$ in $\mathcal{H}ilb_{10}^G(\mathbb{P}^4_k)$ were contained in a component different from $\mathcal{H}ilb_{10}^{G,gen}(\mathbb{P}^4_k)$, then each smoothable $X \in \mathcal{Z}_4$, if any, would be obstructed.

In [I–E] the authors asserted the existence of such a scheme but their computations were affected by a mistake pointed out to the authors by R. Buchweitz in a private communication. In Example 4.1 of [Ia3] the author claimed the smoothability of all points in $\mathcal{Z}_N$ without providing any proof for this. We will give here a quick proof of this fact.

Proposition 4.3. *There exists an unobstructed $X \in \mathcal{Z}_4 \cap \mathcal{H}ilb_{10}^G(\mathbb{P}^4_k)$.*

Proof. Consider the ideal

$$J := (x_3x_4, x_2x_4, x_1x_4, x_1^2 + x_2^2, x_1x_2 + x_3^2, x_1x_3, x_4^3 - b^2x_4 + (b-1)x_1^3, x_3^3, x_2^2x_3, x_2^3)$$

in $k[b, x_1, x_2, x_3, x_4]$. Let $\mathcal{A} := k[b, x_1, x_2, x_3, x_4]/J$ and denote by $\mathcal{X} \to \mathbb{A}^1_k$ the corresponding family.

If $b \neq 0$, then $J = J_1 \cap J_2$ where

$$J_1 := (x_4^2, x_3x_4, x_2x_4, x_1x_4, x_1x_3, x_1x_2 + x_3^2, x_1^2 + x_2^2, x_3^3, x_2^2x_3, x_2^3, bx_2x_3^2 - x_2x_3^2 - b^2x_4),$$
$$J_2 = (x_1, x_2, x_3, x_4^2 - b^2)$$

(one can use any computer algebra system for checking such an equality). Since we have $bx_2x_3^2 - x_2x_3^2 - b^2x_4 \in J_1$, when $b \neq 0$ we have an isomorphism

$$k[x_1, x_2, x_3, x_4]/J_1 \cong k[x_1, x_2, x_3]/(x_1x_3, x_1x_2 + x_3^2, x_1^2 + x_2^2) \cong A^{1,0}_{3,3,8}.$$

Such an algebra is smoothable by Lemma 3.4. Since the fibres $\mathcal{X}_b$ with $b \neq 0$ are union of $\operatorname{spec}(A^{1,0}_{3,3,8})$ with two simple points, they are smoothable too. Moreover their degree is 10, thus they are in $\mathcal{H}ilb_{10}^{G,gen}(\mathbb{P}^4_k)$. When $b = 0$, the special fibre $X := \mathcal{X}_0$ is defined in $k[x_1, x_2, x_3, x_4]$ by the homogeneous ideal

$$I := (x_3x_4, x_2x_4, x_1x_4, x_1^2 + x_2^2, x_1x_2 + x_3^2, x_1x_3, x_4^3 - x_1^3, x_3^3, x_2^2x_3, x_2^3).$$

Hence it is irreducible since it is supported only on the point $[1,0,0,0,0] \in \mathbb{P}^4_k$. Moreover the corresponding algebra $A := k[x_1, x_2, x_3, x_4]/I \cong \mathcal{A}_0$ has Hilbert function $h_A = (1,4,4,1)$ and it is easy to check that its socle is generated by x_1^3, thus $X \in \mathcal{Z} \subseteq \mathcal{H}ilb_{10}^G(\mathbb{P}^4_k)$.

We conclude that, in order to prove the irreducibility of $\mathcal{H}ilb_{10}^{G}(\mathbb{P}^4_k)$, it suffices to check that $X \notin \mathrm{Sing}(\mathcal{H}ilb_{10}^{G}(\mathbb{P}^4_k))$. Since $\dim(\mathcal{H}ilb_{10}^{G,gen}(\mathbb{P}^4_k)) = 40$ it suffices to check that the tangent space at the point $X \in \mathcal{H}ilb_{10}^{G}(\mathbb{P}^4_k)$, which is canonically identified with $H^0\big(X, \mathcal{N}_X\big)$, has dimension 40.

In our case it suffices to check that $\deg(X^{(2)}) = \dim_k(k[x_1, x_2, x_3, x_4]/I^2) = 50$, thanks to Formula (2.6), and this can be computed via any computer software for symbolic computation. This computation concludes the proof of the statement. $\square$

Now assume $N \geq 5$ and let $X \in \mathcal{Z}_N$. Due to the definition of $\mathcal{Z}_N$ we know that there is an embedding $X \subseteq \mathbb{P}^4_k$. Thanks to the discussion above we know that X is smoothable in $\mathbb{P}^4_k$, thus the same holds in $\mathbb{P}^N_k$ due to Lemma 2.2. This proves the following

Corollary 4.4. *Let $X \cong \mathrm{spec}(A) \subseteq \mathbb{P}^N_k$, where $h_A = (1, 4, 4, 1)$ and $N \geq 4$. Then X is smoothable in $\mathbb{P}^N_k$.* $\square$

We are now ready to summarize the results proved in this section and in the previous one in order to give the

Proof of the Main Theorem. In order to prove that $\mathcal{H}ilb_{10}^{G}(\mathbb{P}^N_k)$ is irreducible it suffices to check $\mathcal{H}ilb_{10}^{G}(\mathbb{P}^N_k) = \mathcal{H}ilb_{10}^{G,gen}(\mathbb{P}^N_k)$, i.e. that each Gorenstein subscheme $X \subseteq \mathbb{P}^N_k$ of dimension 0 is smoothable.

If X has at least two components this follows from Proposition 2.7. Thus we restrict our attention to irreducible schemes X. Let $X \cong \mathrm{spec}(A)$ for some local Artinian Gorenstein k–algebra with maximal ideal $\mathfrak{M}$. If $\dim_k(\mathfrak{M}^2 / \mathfrak{M}^3) = 1$, then the smoothability of X is proven in Proposition 3.1, if $\dim_k(\mathfrak{M}^2 / \mathfrak{M}^3) = 2$, in Propositions 3.2 and 3.3, if $\dim_k(\mathfrak{M}^2 / \mathfrak{M}^3) = 3$ in Propositions 3.4 and 3.5, if $\dim_k(\mathfrak{M}^2 / \mathfrak{M}^3) = 4$ in Corollary 4.4. $\square$

Lemma 6.21 of [I–K] essentially asserts the reducibility of $\mathcal{H}ilb_{14}^{G}(\mathbb{P}^N_k)$ when $N \geq 6$. Indeed the authors claim the existence of a scheme $X \cong \mathrm{spec}(A) \subseteq \mathbb{P}^6_k$, where $h_A = (1, 6, 6, 1)$ and having tangent space of dimension 76.

Since the main component $\mathcal{H}ilb_{14}^{G,gen}(\mathbb{P}^N_k) \subseteq \mathcal{H}ilb_{14}^{G}(\mathbb{P}^N_k)$ has dimension 84 we infer the existence of a second component $\mathcal{H} \subseteq \mathcal{H}ilb_{14}^{G}(\mathbb{P}^N_k)$ of dimension at most 76.

In order to construct such a scheme it suffices to make again use of the theory of inverse systems explained above. For example, if one considers $N = 6$ and the polynomial

$$\begin{aligned} g(y_1, \dots, y_6) := y_1^3 + y_2^3 + y_3^3 + y_4^3 + y_5^3 + y_6^3 + (y_1 + y_2 + y_3 + y_4 + y_5 + y_6)^3 + \\ +(2y_1 + y_2 - 2y_3 + y_5 - y_6)^3 + (-y_1 - 2y_2 - 2y_3 - 2y_4 + 2y_5 - 2y_6)^3 + \\ + (-y_1 - y_2 + 2y_3 + y_4 - 2y_6)^3 \end{aligned}$$

then an explicit computation shows that the corresponding local, Artinian, Gorenstein k–algebra A has $h_A = (1, 6, 6, 1)$ and, using Formula (2.6), that

$$h^0\big(X, \mathcal{N}_X\big) = \dim_k(k[x_1, \dots, x_6]/(g^{\perp})^2) - \dim_k(k[x_1, \dots, x_6]/g^{\perp}) = 76.$$

No analogous results are known for $\mathcal{H}ilb_d^{G,gen}(\mathbb{P}^N_k)$ with $11 \le d \le 13$. Similar computations with $N=5$ and polynomials of degree 3, give at most local, Artinian, Gorenstein k–algebras A with $h_A=(1,5,5,1)$ such that $X=\operatorname{spec}(A)\subseteq \mathbb{P}^6_k$ satisfies $h^0\big(X,\mathcal{N}_X\big)=60$ which is exactly the dimension of $\mathcal{H}ilb_{12}^{G,gen}(\mathbb{P}^5_k)$.

For this reason we explicit the following question essentially due to A.V. Iarrobino.

Question 4.5. Is $\mathcal{H}ilb_d^{G}(\mathbb{P}^N_k)$ irreducible if and only if $d\le 13$?

5. The singular locus of $\mathcal{H}ilb_{10}^{G}(\mathbb{P}^N_k)$

In this last section, we describe the singular locus of $\mathcal{H}ilb_{10}^{G}(\mathbb{P}^N_k)$. Since $\mathcal{H}ilb_{d}^{G}(\mathbb{P}^N_k)$ is irreducible of dimension dN for $d\le 10$, it follows that X is obstructed, i.e. it is singular in $\mathcal{H}ilb_{d}^{G}(\mathbb{P}^N_k)$, if and only if $h^0\big(\mathbb{P}^N_k,\mathcal{N}_X\big)>dN$.

Due to Formula (2.4) and Proposition 2.5, this can happen only when there is an irreducible component $Y\subseteq X$ of degree d in the following list:

(1) $Y\cong\operatorname{spec}(A_{n,d})$, with $6\le n+2\le d$;
(2) $Y\cong\operatorname{spec}(A^t_{n,2,d})$ with $t=1,2$ and $8\le n+4\le d$;
(3) $Y\cong\operatorname{spec}(A^{t,\alpha}_{n,3,n+5})$ with $t=1,\dots,6$ and $9\le n+5=d$ (if $n=5$ then $Y=X$);
(4) $Y=X\cong\operatorname{spec}(A^t_{4,3,10})$ with $t=0,\dots,6$;
(5) $Y=X\cong\operatorname{spec}(A^t_{4,2,2,10})$ with $t=1,2,3$;
(6) $Y=X\in\mathcal{Z}_N$.

In Section 5 of [C–N2] we checked that in cases (1) and (2), the corresponding schemes are obstructed. In case (3) it is proven there that Y is obstructed if and only if $t=4,5,6$ when $n=4$, by computing explicitly $h^0\big(Y,\mathcal{N}_Y\big)$ where the embedding $Y\subseteq\mathbb{A}^n_k\subseteq\mathbb{P}^n_k$ is the natural one corresponding to the representation of Y as spectrum of a quotient of $k[x_1,\dots,x_n]$ and making use of Formula (2.6) as already done above. We now examine with the same approach, using any computer software for symbolic calculations, the cases (3) with $n=5$ and (4), (5) with $n=4$.

In case (3) we have that the normal sheaf $\mathcal{N}_X$ of the embedding induced by the natural quotient $k[x_1,\dots,x_5]\twoheadrightarrow A^{t,\alpha}_{n,3,n+5}$ satisfies

$$h^0\big(X,\mathcal{N}_X\big)=\begin{cases}57 & \text{if } t=1,2,3,\\ 64 & \text{if } t=4,5,6.\end{cases}$$

In case (4) with respect to the natural quotient $k[x_1,\dots,x_4]\twoheadrightarrow A^t_{4,3,10}$ we have

$$h^0\big(X,\mathcal{N}_X\big)=\begin{cases}40 & \text{if } t=0,1,\\ 45 & \text{if } t=2,3,4,5,6.\end{cases}$$

Finally, in case (5), with respect to the natural quotient $k[x_1,\dots,x_4]\twoheadrightarrow A^t_{4,2,2,10}$ we have

$$h^0\big(X,\mathcal{N}_X\big)=45\quad \text{if } t=1,2,3.$$

We can summarize the above results in the following

Theorem 5.3. *Let $X \in \mathcal{H}ilb^G_{10}(\mathbb{P}^N_k) \setminus \mathcal{Z}_N$. Then X is obstructed if and only if it contains an irreducible component isomorphic to either* $\mathrm{spec}(A_{n,d})$ *or* $\mathrm{spec}(A^t_{n,2,d})$*, where $n \geq 4$, or* $\mathrm{spec}(A^{t,\alpha}_{4,3,9})$*, where $t = 4, 5, 6$, or* $\mathrm{spec}(A^t_{4,3,10})$*, where $t = 2, 3, 4, 5, 6$, or* $\mathrm{spec}(A^{t,\alpha}_{5,3,10})$ *or* $\mathrm{spec}(A^t_{4,2,2,10})$*, without restrictions on t.* □

It is natural to ask what happens in the case $X \in \mathcal{Z}_N \subseteq \mathcal{H}ilb^G_{10}(\mathbb{P}^N_k)$. We checked in the previous section that the general scheme in $\mathcal{Z}_N$ is not obstructed. In principle the theory of inverse system and the classification of cubic surfaces (e.g. as the one in [B–L]) could allow us to complete the description of points in $\mathcal{Z}_N$, hence it could help to describe completely the singular locus of $\mathcal{H}ilb^G_{10}(\mathbb{P}^N_k)$.

Unfortunately, taking into account of [B–L], we have at least 22 different cases to handle, most of them depending on many parameters. Thus a direct approach seems to be useless in this case. Thus we need another method. Notice that each point in $\mathcal{Z}_N$ corresponds to a local Artinian Gorenstein k–algebra A with Hilbert function $(1, 4, 4, 1)$.

As explained in the previous section, such kind of algebra is naturally graded, i.e. it can be written as a suitable quotient quotient $S := k[x_1, x_2, x_3, x_4]/I$ with I homogeneous and it corresponds, via Macaulay's correspondence, to a cubic form g, i.e. $I = g^\perp$.

Lemma 5.5. *The minimal free resolution of $A \cong S/g^\perp$ over S has the form*

$$0 \longrightarrow S(-7) \longrightarrow S^6(-5) \oplus S^\beta(-4) \longrightarrow S^{5+\beta}(-4) \oplus S^{5+\beta}(-3) \longrightarrow$$
$$\longrightarrow S^\beta(-3) \oplus S^6(-2) \longrightarrow S \longrightarrow A \longrightarrow 0$$

for some $\beta \geq 0$.

Proof. The ideal $g^\perp$ has obviously six minimal generators of degree 2, but it could also have some more minimal generators in degree 3 or higher. Thus the minimal free resolution of A over S ends with

$$S^6(-2) \oplus S(-3)^\beta \oplus F \longrightarrow S \longrightarrow A \longrightarrow 0$$

where $\beta \geq 0$ is the number of minimal cubic generators of $g^\perp$ and F is a direct sum of $S(-j)$ with $j \geq 4$.

If F does not contain the direct summand $S(-4)$,then the cubic forms in the ideal $g^\perp$ would generate its degree 4 homogeneous part, thus they would generate $g^\perp$ in degree greater than 3, i.e. $F = 0$. It remains to examine the case when $g^\perp$ has a minimal generator in degree 4.

Since A is Gorenstein with maximum socle degree 3, it follows that the minimal free resolution is self–dual up to twisting by $S(-7)$ (this is a well–known fact. For the sake of completeness we quote [B–H] as reference: in particular Corollary 3.3.9, Proposition 3.6.11, Examples 3.6.15, Theorem 3.6.19 and the remark after it). Moreover the middle free module cannot contain $S(-2)$ summands since the generators in degree 2 are obviously linearly independent. Combining such remarks we obtain that the minimal free resolution of A has the shape

$$0 \longrightarrow S(-7) \longrightarrow S^6(-5) \oplus S^\beta(-4) \oplus \check{F}(-7) \longrightarrow G \longrightarrow$$
$$\longrightarrow S^\beta(-3) \oplus S^6(-2) \oplus F \longrightarrow S \longrightarrow A \longrightarrow 0$$

On one hand, by assumption, $S(-4)$ is a free addendum of F, hence $S(-3)$ is a free addendum of $\check{F}(-7)$. On the other hand the resolution above is minimal, thus at each step the minimal degree of the syzygies must increase at least by one. This two remarks yields a contradiction, thus $F = 0$.

For the same reasons G contains only direct summand of the form $S(-j)$ with $j \geq 3$ and $G \cong \check{G}(-7)$. A simple computation thus yields $G \cong S^{5+\beta}(-4) \oplus S^{5+\beta}(-3)$. $\square$

Remark 5.6. Notice that the argument above can be also used for proving the following assertion. Let $I \subseteq k[x_1, \dots, x_N]$ be a homogeneous ideal such that $A := k[x_1, \dots, x_N]/I$ is a local Artinian Gorenstein k–algebra with maximum socle degree e. Then I has a minimal generator in degree $e+1$ if and only if $A \cong k[t]/(t^{e+1})$ or, equivalently, if and only if $I = g^{\perp}$ with $g = \ell^{e+1}$ for some linear form $\ell \in k[y_1, \dots, y_N]$.

At this point we are ready to start with our classifications results. We first examine the general case.

Proposition 5.7. *Using the notation above let $A^{(2)} := S/(g^{\perp})^2$. If $\beta = 0$ in Lemma 5.5, then $h_{A^{(2)}} = (1, 4, 10, 20, 14, 1)$.*

Proof. Let $f_1, \dots, f_6 \in S_2$ be a minimal set of quadratic generators of $g^{\perp}$. Since the ring A is Artinian, we can assume that $f_1, \dots, f_4$ is a regular sequence in S. To fix notation, we assume that the first map $\varphi \colon S^6(-2) \to S$ of the resolution in Lemma 5.5 of A is given by $\varphi(e_i) = f_i$ for $i = 1, \dots, 6$, where $e_1, \dots, e_6$ is the canonical basis of $S^6(-2)$.

Let $M := (M_1|M_2)$ be the matrix representing the map $S^5(-4) \oplus S^5(-3) \to S^6(-2)$ with respect to the canonical bases of the involved free modules. Trivially the elements of M_1 have degree 2 while the ones of M_2 have degree 1. Let $V \subseteq S_1$ (resp. $W \subseteq S_1$) be the subspace generated by the elements of the 5–th row (resp. 6–th row) of M_2. If $\dim_k(V) \leq 2$ and $\dim_k(W) \leq 2$, then we can obtain a degree 1 syzygy of $g^{\perp}$ with the last two entries equal to 0, that is to say, there exists a degree 1 syzygy of $f_1, \dots, f_4$, a contradiction, since the resolution of $I = (f_1, \dots, f_4) \subseteq S$ is Koszul being $f_1, \dots, f_4$ a regular sequence. Hence, either V or W has dimension at least 3. Up to exchange f_5 and f_6, we can finally assume $\dim_k(W) \geq 3$.

The minimal free resolution of $B := S/I$ is

$$0 \longrightarrow S(-8) \longrightarrow S^4(-6) \longrightarrow S^6(-4) \longrightarrow S^4(-2) \longrightarrow S \longrightarrow B \longrightarrow 0,$$

whence $h_B = (1, 4, 6, 4, 1)$. Since $I \subseteq g^{\perp}$, it follows the existence of a natural epimorphism $B \twoheadrightarrow A$ with kernel $g^{\perp}/I$.

Of course, the classes of f_5 and f_6 mod I are in B_2. It is then obvious that $f_5 S_d \subset I$ and $f_6 S_e \subset I$ for some integers d, e. Let $J := (f_1, \dots, f_5)$.

We first consider the case $\dim_k(W) = 4$, i.e. $W = S_1$. In this case $f_6 S_1 \subset J$. From the above inclusion and the short exact sequence

$$0 \longrightarrow g^{\perp}/J \longrightarrow S/J \longrightarrow A \longrightarrow 0$$

we deduce $h_{S/J} = (1, 4, 5, 1)$. Consider now the exact sequence

$$0 \longrightarrow J/I \longrightarrow S/I \longrightarrow S/J \longrightarrow 0.$$

By computing the dimensions of the homogeneous pieces, we obtain $\dim_k((J/I)_j) = 1, 3, 1$, for $j = 2, 3, 4$, respectively, and 0 otherwise. Hence, there exists $\ell_1 \in S_1$ such that $\ell_1 f_5 \in I$, and, if $\ell_1, \dots, \ell_4$ is a basis of S_1, we infer that the cosets of $\ell_2 f_5, \ell_3 f_5, \ell_4 f_5$ are linearly independent in S/I.

Looking at the matrix M_2, after reducing its columns by elementary operations, we can say that there is one column whose last two entries are $\ell_1, 0$, respectively. After reducing the columns of M by elementary operations, all the elements of the 5–th row of M_1 are non-zero. Hence, there are 5 linearly independent elements in $(\ell_2, \ell_3, \ell_4)^2$ which are in I when multiplied by f_5. Since there are no minimal syzygies in degree 3 and $(\ell_2, \ell_3, \ell_4)^3 f_5 \subseteq I$, we can choose generators of (ℓ_2, ℓ_3, ℓ_4) in such a way that $\ell_2^2 f_5, \ell_2\ell_3 f_5, \ell_3^2 f_5, \ell_2\ell_4 f_5, \ell_4^2 f_5 \in I$, while the coset of $\ell_3\ell_4 f_5$ spans J/I in degree 4.

The ideals $I, J, g^\perp$ give rise to the following sequence of strict inclusions

$$I^2 \subset IJ \subset J^2 \subset Jg^\perp \subset (g^\perp)^2$$

that we will use in order to compute $h_{A^{(2)}}$.

To start with, we consider $B^{(2)} = S/I^2$. On one hand it fits into the exact sequence

$$0 \longrightarrow I/I^2 \longrightarrow B^{(2)} \longrightarrow B \longrightarrow 0.$$

On the other hand $I/I^2 = I \otimes_S S/I \cong (S/I)^4(-2)$, since I is generated by a regular sequence of quadratic forms. Hence $h_{B^{(2)}} = (1, 4, 10, 20, 25, 16, 4)$.

The module IJ/I^2 is generated by the cosets of $f_1 f_5, \dots, f_4 f_5$. Let $a_1, \dots, a_4 \in S$ be such that $a_1 f_1 f_5 + \cdots + a_4 f_4 f_5 \in I^2$. Hence, $(a_1 f_5, \dots, a_4 f_5)$ is zero in $(S/I)^4(-2)$, i.e. $a_i f_5 \in I$ for each $i = 1, \dots, 4$. Thanks to the previous discussion, this happens if, and only if $a_1 = 0$ (when $\deg(a_i) = 0$), $a_i \in (\ell_1)$ (when $\deg(a_i) = 1$), $a_i \in (\ell_1, \ell_2^2, \ell_2\ell_3, \ell_3^2, \ell_2\ell_4, \ell_4^2)$ (when $\deg(a_i) = 2$) and, finally, $a_i \in S_j$ (when $\deg(a_i) = j \geq 3$). Hence $\dim_k((IJ/I^2)_j) = 4, 12, 4$, for $j = 4, 5, 6$ respectively, and 0 otherwise, thus $h_{S/IJ} = (1, 4, 10, 20, 21, 4)$.

Now, consider $C^{(2)} := S/J^2$ and the exact sequence

$$0 \longrightarrow J^2/IJ \longrightarrow S/IJ \longrightarrow C^{(2)} \longrightarrow 0.$$

The module J^2/IJ is generated by the coset of f_5^2, and the assertion $af_5^2 \in IJ$ is equivalent to the assertion $af_5 \in I$. It thus follows from the above discussion and from the computation of $h_{S/IJ}$ we get that $\dim_k((J^2/IJ)_j) = 1, 3$, for $j = 4, 5$, respectively, and 0 otherwise. Hence $h_{C^{(2)}} = (1, 4, 10, 20, 20, 1)$.

The module $Jg^\perp/J^2$ is generated by the cosets of $f_1 f_6, \dots, f_5 f_6$, thus the dimensions of its homogeneous pieces are $\dim_k((Jg^\perp/J^2)_j) = 5$ if $j = 4$, and 0 otherwise, since $f_6 S_1 \subseteq J$. Hence the the Hilbert function of $S/Jg^\perp$ can be computed by using the exact sequence

$$0 \longrightarrow Jg^\perp/J^2 \longrightarrow C^{(2)} \longrightarrow S/Jg^\perp \to 0.$$

We obtain $h_{S/Jg^\perp} = (1, 4, 10, 20, 15, 1)$.

Finally, the module $(g^\perp)^2/Jg^\perp$ is generated by the coset of f_6^2 and it is non-zero only in degree 4. The Hilbert function of $A^{(2)}$ is then equal to $h_{A^{(2)}} = (1, 4, 10, 20, 14, 1)$ as it comes from considering the exact sequence

$$0 \longrightarrow (g^\perp)^2/Jg^\perp \longrightarrow S/Jg^\perp \longrightarrow A^{(2)} \longrightarrow 0.$$

Thus the statement is proved under the extra hypothesis $\dim_k(W) = 4$.

Now, we consider the case $\dim_k(W) = 3$. Of course, up to exchanging the roles of f_5 and f_6, we can also assume $\dim_k(V) \leq 3$. We can reduce the matrix M_2 by using elementary operations on its columns, and so we can assume that two entries of the 6–th row of M_2 are equal to 0. Moreover, from the three non–zero entries of the row, we deduce that $\ell f_6 \in J$ for each $\ell \in W$ and that the last two columns of M_2 have two linearly independent elements on the 5–th row.

Recall that $\dim_k(V)$ is either 2 or 3. In the former case we can assume that, for each column of M_2, if the element of the 5–th row is non–zero, then the element on the 6–th row is zero and conversely. In the latter case we can assume that the previous situation happens on 4 columns of M_2. Furthermore, if we reduce the matrix M by using elementary operations on its columns, not all the entries of the 6–th row of M_1 can be equal to 0, due to the fact that $f_6 S_e \subseteq I$.

Then, if $S_1 = W \oplus \langle \ell \rangle$, we can assume that $\ell^2 f_6 \in J$. Let $C = S/J$ and consider the short exact sequence

$$0 \longrightarrow g^\perp/J \longrightarrow S/J \longrightarrow A \longrightarrow 0$$

From the discussion above, we deduce that $h_{S/J}(1) = 4$ and $h_{S/J}(2) = 5$. Moreover, $\ell f_6 \notin J$ and so $h_{S/J}(3) = 2$, but $h_{S/J}(4) = 0$, since $\ell^2 f_6 \in J$. Hence $h_{S/J} = (1, 4, 5, 2)$. We can also consider the short exact sequence

$$0 \longrightarrow J/I \longrightarrow S/I \longrightarrow S/J \longrightarrow 0.$$

Thus the Hilbert function of J/I satisfies $\dim_k((J/I)_j) = 1, 2, 1$ for $j = 2, 3, 4$ respectively, and $\dim_k((J/I)_j) = 0$ otherwise.

From the analysis of the elements of the 5–th row of M corresponding to the 0 entries on the last row of M, we get that there exists a dimension 2 subspace $V' \subseteq S_1$ such that $\ell f_5 \in I$ for each $\ell \in V'$. Let us choose $V'' \subseteq S_1$ such that $S_1 = V' \oplus V''$. Let ℓ_1, ℓ_2 be a basis of V''. Then J/I is generated by the coset of f_5 in degree 2 and by the cosets of $\ell_i f_5, i = 1, 2$, in degree 3. Furthermore, we have that two among $\ell_1^2 f_5, \ell_1\ell_2 f_5, \ell_2^2 f_5$ are in I. The columns of M_2 have degree 2, and so $\ell_i^2 f_5 \in I$, $i = 1, 2$, since $f_5(\ell_1, \ell_2)^3 \subseteq I$, but we have no minimal syzygies in degree 3.

As in the case $\dim_k(W) = 4$, the ideals I, J and $g^\perp$ give rise to the following sequence of strict inclusions

$$I^2 \subset IJ \subset J^2 \subset Jg^\perp \subset (g^\perp)^2$$

that we will use again to compute $h_{A^{(2)}}$. The Hilbert function of $B^{(2)}$ has been already computed above, and we do not repeat the computation.

The module IJ/I^2 is generated by the cosets of $f_1f_5, f_2, f_5, f_3f_5, f_4f_5$ and fits into the short exact sequence

$$0 \longrightarrow IJ/I^2 \longrightarrow B^{(2)} \longrightarrow S/IJ \longrightarrow 0.$$

Let $a_1, \dots, a_4 \in S$ be such that $\sum_{i=1}^4 a_i f_i f_5 \in I^2$. Then $(a_1f_5, \dots, a_4f_5)$ is zero in $(S/I)^4(-2)$, i.e. $a_if_5 \in I$ for each $i = 1, \dots, 4$. If $\deg(a_i) = 0$, this implies $a_i = 0$, for each i; if $\deg(a_i) = 1$, we get $a_i \in V'$ for each i; if $\deg(a_i) = 2$, then $a_i \in V'S_1 + \langle \ell_1^2, \ell_2^2\rangle$; finally, if $\deg(a_i) \geq 3$, then $a_if_5 \in I$ for each i. It follows that $\dim_k((IJ/I^2)_j) = 4, 8, 4$, for $j = 4, 5, 6$ respectively and 0 otherwise. Hence $h_{S/IJ} = (1, 4, 10, 20, 21, 8)$.

The next step consists in considering the short exact sequence

$$0 \longrightarrow J^2/IJ \longrightarrow S/IJ \longrightarrow C^{(2)} \longrightarrow 0.$$

The module J^2/IJ is generated by the coset of f_5^2. We know that $S_j = (IJ)_j$ for $j \geq 6$, hence it is enough to consider $a \in S$ such that $af_5^2 \in IJ$, with $\deg(a) \leq 1$. This means that $af_5 \in I$, and so either $a = 0$ (when $\deg(a) = 0$) or $a \in V'$(when $\deg(a) = 1$). It follows that $\dim_k((J^2/IJ)_j) = 1, 2$, for $j = 4, 5$ respectively, and 0 otherwise. Hence, the Hilbert function of $C^{(2)}$ is $h_{C^{(2)}} = (1, 4, 10, 20, 20, 6)$.

The module IJ/J^2 is generated by the cosets of $f_1f_6, \dots, f_5f_6$. Then, we have that $\dim_k((Jg^\perp/J^2)_4) = 5$. Let $a \in S_1$, and consider af_5f_6. If $a \in W$, then $af_6 \in J$, and so $af_5f_6 \in Jg^\perp$. If $a \in V'$, then $af_5 \in I$, and so $af_5f_6 \in (f_1f_6, \dots, f_4f_6)$. Hence, if $W + V' = S_1$, we deduce that $(Jg^\perp/J^2)_5$ is spanned by the cosets of $\ell f_1f_6, \dots, \ell f_4f_6$, whence $\dim_k((Jg^\perp/J^2)_5) = 4$. If $W \supset V'$, then the cosets of $\ell f_1f_6, \dots, \ell f_5f_6$ are linearly independent, thus $\dim_k((Jg^\perp/J^2)_5) = 5$. Hence, the Hilbert function of $S/Jg^\perp$ is either $h_{S/Jg^\perp} = (1, 4, 10, 20, 15, 2)$ (when $V' \not\subseteq S_1$) or $h_{S/Jg^\perp} = (1, 4, 10, 20, 15, 1)$, (when $V' \subset W$), as we easily obtain from the short exact sequence

$$0 \longrightarrow Jg^\perp/J^2 \longrightarrow C^{(2)} \longrightarrow S/Jg^\perp \longrightarrow 0.$$

In both the cases, $(g^\perp)^2/Jg^\perp$ is generated by the coset of f_6^2, hence $\dim_k(((g^\perp)^2/Jg^\perp)_4) = 1$.

If $V' \not\subseteq W$, then ℓf_6^2 spans $((g^\perp)^2/Jg^\perp)_5$ as vector space, thus $\dim_k(((g^\perp)^2/Jg^\perp)_5) = 1$. From the exact sequence

$$0 \longrightarrow (g^\perp)^2/Jg^\perp \longrightarrow S/Jg^\perp \longrightarrow A^{(2)} \longrightarrow 0$$

we finally obtain that $h_{A^{(2)}} = (1, 4, 10, 20, 14, 1)$. If $V' \subset W$, then we certainly have $h_{A^{(2)}} = (1, 4, 10, 20, 15, 1) - (0, 0, 0, 0, 1, h_5) = (1, 4, 10, 20, 14, 1 - h_5)$ where $h_5 \geq 0$. Due to the Main Theorem then $\mathcal{H}ilb_{10}^G(\mathbb{P}_k^4)$ is irreducible, thus the scheme $X := \operatorname{spec}(A)$ embedded in $\mathbb{A}_k^4 \subseteq \mathbb{P}_k^4$ via the natural quotient $S \twoheadrightarrow A$ lies in a scheme of dimension 40. Proposition 2.5 thus yields that

$$40 \leq h^0\big(X, \mathcal{N}_X\big) = \dim_k(A^{(2)}) - \dim_k(A) = 40 - h_5,$$

whence $h_5 = 0$. We conclude that $h_{A^{(2)}} = (1, 4, 10, 20, 14, 1)$ also in this second case. $\square$

Now we examine the case when $\beta \geq 1$. In this case $g^\perp$ has at least one minimal cubic generator. By [C-R-V], Theorem 6.18, there exists $\ell \in S_1$ such that $\ell(g) \in R_2$ is a rank 1 quadric. Up to a change of coordinates, we can assume $\ell = x_4$, and $x_4(f) = y^2$ for some $y = \sum_{i=1}^4 b_i y_i \in R_1$. Either $b_4 \neq 0$ or $b_4 = 0$.

In the former case we can assume that $b_4 = 1$. If $b_1y_1 + b_2y_2 + b_3y_3 = 0$, then $g = y_4^3 + g_0$ for a suitable $g_0 \in k[y_1, y_2, y_3]$. If $b_1y_1 + b_2y_2 + b_3y_3$ is non–zero, then, up to a change of variables, we have $g = y_4^3 + y_4^2y_2 + y_4y_2^2 + g_1$ for a suitable cubic form $g_1 \in k[y_1, y_2, y_3]$. By setting $x_2 = X_4 - X_2, x_i = X_i$ for $i = 1, 3, 4$, and $y_4 = Y_4 + Y_2$, $y_2 = -Y_2$, $y_i = Y_i$ for $i = 3, 4$, then $g = Y_4^3 + Y_2^3 + g_2(-Y_1, Y_2, Y_3)$.

In the latter case we have that $b_4 = 0$. Necessarily $b_1y_1 + b_2y_2 + b_3y_3 \neq 0$, hence up to a proper change of the variables we can assume $g = y_3^2y_4 + \widehat{g}(y_1, y_2, y_3)$.

The above discussion proves the "only if" of the following

Lemma 5.8. *Let $g \in S_3$. Then, $g^\perp$ has minimal generators in degree 3 if and only if there exists a cubic form $\widehat{g} \in k[y_1, y_2, y_3]$ such that, up to a proper choice of coordinates in R, either $g = y_4^3 + \widehat{g}$ or $g = y_3^2y_4 + \widehat{g}$.*

Proof. It remains to prove the "if" part. If either $g = y_4^3 + \widehat{g}$ or $g = y_3^2y_4 + \widehat{g}$ for some cubic form $\widehat{g} \in k[y_1, y_2, y_3]$, then $x_4(g)$ is equal either to $3y_4^2$ or to $2y_3^2$, hence $x_4(g)$ is a rank 1 quadric. Again by [C-R-V], Theorem 6.18, $g^\perp$ has a minimal generator in degree 3. $\square$

We now go to complete our classification.

Proposition 5.9. *Using the notation above let $A^{(2)} := S/(g^\perp)^2$. If $\beta \geq 1$ in Lemma 5.5, then either $\beta = 1$ and $h_{A^{(2)}} = (1, 4, 10, 20, 14, 1)$ or $\beta = 3$ and $h_{A^{(2)}} = (1, 4, 10, 20, 16, 4)$.*

Proof. Due to Lemma 5.8 we can assume that either $g = y_4^3 + \widehat{g}$ or $g = y_3^2y_4 + \widehat{g}$ for some cubic form $\widehat{g} \in k[y_1, y_2, y_2]$.

Consider the first case. Up to a suitable change of coordinates, $\widehat{g}$ is equal to one of the following

$$y_3^3, \qquad y_2y_3^2, \qquad y_2y_3(y_2 - y_3), \qquad y_1y_2y_3, \qquad y_3(y_1y_3 - y_2^2), \qquad y_2(y_1y_3 - y_2^2),$$
$$y_1^2y_3 + y_2^2y_3 - y_2^3, \qquad y_1^2y_3 - y_2^3, \qquad y_1^2y_3 - y_2^3 + (1 + t)y_2^2y_3 - ty_2y_3^2$$

where $t \in k$ is different from 0 and 1. In the various cases we perform the computation using any computer software for symbolic calculations, and we report the results.

The first three choices give Artinian Gorenstein rings with Hilbert function different from $(1, 4, 4, 1)$, because g is a cone in those cases.

If $g = y_4^3 + y_1y_2y_3$, then

$$g^\perp = (x_1^2, x_2^2, x_3^2, x_1x_4, x_2x_4, x_3x_4, 6x_1x_2x_3 - x_4^3).$$

Hence, $\beta = 1$, and $h_{A^{(2)}} = (1, 4, 10, 20, 14, 1)$.

If $g = y_4^3 + y_3(y_1y_3 - y_2^2)$, then

$$g^\perp = (x_1^2, x_1x_2, x_2^2 + x_1x_3, x_1x_4, x_2x_4, x_3x_4, 3x_1x_3^2 - x_4^3, x_2x_3^2, x_3^3).$$

Hence, $\beta = 3$, and $h_{A^{(2)}} = (1, 4, 10, 20, 16, 4)$.

If $g = y_4^3 + y_2(y_1y_3 - y_2^2)$, then

$$g^\perp = (x_1^2, x_2^2 + 6x_1x_3, x_3^2, x_1x_4, x_2x_4, x_3x_4, 6x_1x_2x_3 - x_4^3).$$

Hence, $\beta = 1$, and $h_{A^{(2)}} = (1, 4, 10, 20, 14, 1)$.

If $g = y_4^3 + y_1^2y_3 + y_2^2y_3 - y_2^3$, then

$$g^\perp = (x_1^2 - x_2^2 - 3x_2x_3, x_1x_2, x_3^2, x_1x_4, x_2x_4, x_3x_4, 3x_2^2x_3 - x_4^3).$$

Hence, $\beta = 1$, and $h_{A^{(2)}} = (1, 4, 10, 20, 14, 1)$.

If $g = y_4^3 + y_1^2y_3 - y_2^3$, then

$$g^\perp = (x_1x_2, x_2x_3, x_3^2, x_1x_4, x_2x_4, x_3x_4, x_1^3, x_2^3 + x_4^3, 3x_1^2x_3 - x_4^3).$$

Hence, $\beta = 3$, and $h_{A^{(2)}} = (1, 4, 10, 20, 16, 4)$.

If $g = y_4^3 + y_1^2y_3 - y_2^3 + (1+t)y_2^2y_3 - ty_2y_3^2$, then

$$\begin{aligned} g^\perp = (x_1x_2, x_1x_4, &x_2x_4, x_3x_4, t(1+t)x_1^2 - tx_2^2 + 3x_3^2, \\ &(t^2 - t + 1)x_1^2 - (1+t)x_2^2 - 3x_2x_3, x_1^3, x_1x_3^2, x_3^3, x_2^3 + x_4^3, \\ &tx_1^2x_3 + x_2x_3^2, (1+t)x_1^2x_3 - x_2^2x_3, 3x_1^2x_3 + x_2^3). \end{aligned}$$

If $t^2 - t + 1 \neq 0$, then $\beta = 1$, and $h_{A^{(2)}} = (1, 4, 10, 20, 14, 1)$. If $t^2 - t + 1 = 0$, then $\beta = 3$ and $h_{A^{(2)}} = (1, 4, 10, 20, 16, 4)$. By the way, it is well–known that the condition $t^2 - t + 1 = 0$ corresponds to the j–invariant of the smooth cubic to be 0 (see [Ha], Section IV.4).

Let us consider now the second case, i.e. $g = y_3^2y_4 + \widehat{g}$ for some cubic form $\widehat{g} \in k[y_1, y_2, y_2]$. With a change of coordinates, we can assume that

$$g = y_3^2y_4 + y_3(b_1y_2^2 + 2b_2y_1y_2 + b_3y_1^2) + (b_4y_2^3 + b_5y_1y_2^2 + b_6y_1^2y_2 + b_7y_1^3).$$

The form $b_4y_2^3 + b_5y_1y_2^2 + b_6y_1^2y_2 + b_7y_1^3$ in the expression of g can have either three simple roots, or a triple root, or a simple root and a double one. According to its roots, up to a change of coordinates, it can be written as either $y_1^3 + y_2^3$, or y_2^3, or $y_1y_2^2$. Accordingly g has one of the following forms:

$$y_3^2y_4 + y_3(b_1y_2^2 + 2b_2y_1y_2 + b_3y_1^2) + y_1^3 + y_2^3, \qquad y_3^2y_4 + y_3(b_1y_2^2 + 2b_2y_1y_2 + b_3y_1^2) + y_2^3,$$
$$y_3^2y_4 + y_3(2b_2y_1y_2 + b_3y_1^2) + y_1y_2^2$$

(in the last case we made the extra change of variables $y_1 \to y_1 + b_1y_3$).

In the first case, we have that

$$\begin{aligned} g^\perp = (&x_4^2, x_2x_4, x_1x_4, x_1x_2 - b_2x_3x_4, 3x_2x_3 - b_2x_1^2 - b_1x_2^2 + (b_1^2 + b_2b_3)x_3x_4,\\ &3x_1x_3 - b_3x_1^2 - b_2x_2^2 + (b_1b_2 + b_3^2)x_3x_4, x_1^3 - 3x_3^2x_4, x_2^3 - 3x_3^2x_4, x_3^3). \end{aligned}$$

If $b_2 \neq 0$, then $\beta = 1$ and $h_{A^{(2)}} = (1, 4, 10, 20, 14, 1)$. If $b_2 = 0$, then $\beta = 3$, and $h_{A^{(2)}} = (1, 4, 10, 20, 16, 4)$.

In the second case, we have that

$$\begin{aligned} g^\perp = (&x_4^2, x_2x_4, x_1x_4, x_1^2 - b_3x_3x_4, x_1x_2 - b_2x_3x_4,\\ &3b_2x_1x_3 - 3b_3x_2x_3 + (b_1b_3 - b_2^2)x_2^2 - b_1(b_1b_3 - b_2^2)x_3x_4,\\ &x_3^3, x_1x_3^2, x_2x_3^2, x_2^3 - 3x_3^2x_4, x_2^2x_3 - b_1x_3^2x_4). \end{aligned}$$

If $b_2 = b_3 = 0$, then g is a cone, and so $g^\perp$ is degenerate. Hence, we can assume that either $b_2 \neq 0$, or $b_3 \neq 0$. In both cases, $\beta = 3$, and $h_{A^{(2)}} = (1, 4, 10, 20, 16, 4)$.

In the last case, we have that

$$\begin{aligned} g^\perp = (&x_4^2, x_2x_4, x_1x_4, x_1^2 - b_3x_3x_4, x_2x_3 - b_2x_2^2,\\ &x_1x_3 - b_2x_1x_2 - b_3x_2^2 + b_2^2x_3x_4, x_3^3, x_2^3, x_1x_3^2, x_1x_2^2 - x_3^2x_4). \end{aligned}$$

If $b_3 \neq 0$, then $\beta = 1$ and $h_{A^{(2)}} = (1, 4, 10, 20, 14, 1)$. If $b_3 = 0$, then $\beta = 3$ and $h_{A^{(2)}} = (1, 4, 10, 20, 16, 4)$. $\square$

Now let $g \in R_3$ and $A := S/g^\perp$. Let $X := \operatorname{spec}(A) \subseteq \mathbb{A}^4_k \subseteq \mathbb{P}^4_k$ be the embedding associated to the quotient $k[x_1, x_2, x_3, x_4] \twoheadrightarrow A$. An immediate consequence of Propositions 5.6 and 5.7, of Formula (2.4) and of Proposition 2.5 is that the normal bundle $\mathcal{N}_X$ satisfies

$$h^0\big(X, \mathcal{N}_X\big) = \begin{cases} 40 & \text{if } g \text{ is as in Proposition 5.7,}\\ 45 & \text{if } g \text{ is as in Proposition 5.8.} \end{cases}$$

The same argument used in the proof of Theorem 5.3, thus yields

Theorem 5.9. *Let $g \in R_3$, $A := S/g^\perp$ and $X := \operatorname{spec}(A) \in \mathcal{Z}_N \subseteq \mathcal{H}ilb_{10}^G(\mathbb{P}^N_k)$. The scheme X is obstructed if and only if $\beta = 3$.* $\square$

References

[A–H] J. Alexander, A. Hirschowitz, *Polynomial interpolation in several variables*, J. Algebraic Geom. **4** (1995), 201–222.

[B-L] M. Brundu, A. Logar, *Parametrization of the orbits of cubic surfaces*, Transform. Groups **3** (1998), 209–239.

[B–H] W. Bruns, J. Herzog, *Cohen–Macaulay rings*, Cambridge studies in advanced mathematics 39, Cambridge U.P., 1993.

[C–E–V–V] D.A. Cartwright, D. Erman, M. Velasco, B. Viray, *Hilbert schemes of 8 point in* $\mathbb{A}^d$, math.AG/0803.0341.

[Cs] G. Casnati, *Isomorphism types of Artinian Gorenstein algebras of multiplicity at most* 9 (to appear in Commun. Algebra).

[C–N1] G. Casnati, R. Notari, *On some Gorenstein loci in* $\mathcal{H}ilb_6(\mathbb{P}^4_k)$, J. Algebra **308** (2007), 493–523.

[C–N2] G. Casnati, R. Notari, *On the Gorenstein locus of some punctual Hilbert schemes*, J. Pure Appl. Algebra **213** (2009), 2055-2074.

[C-R-V] A. Conca, M.E. Rossi, G. Valla, *Gröbner flags and Gorenstein algebras*, Compositio Math. **129** (2001), 95–121.

[E–R] J. Elias, M.E. Rossi, *Isomorphism classes of short Gorenstein local rings via Macaulay's inverse system*, arXiv:0911.3565v1 [math.AC] 18 November 2009 (2009).

[E–V] J. Elias, G. Valla, *Isomorphism classes of certain Artinian Gorenstein algebras*, arXiv:0802.0841v3 [math.AC] 26 April 2009 (2009).

[Fo] J. Fogarty, *Algebraic families on an algebraic surface*, Amer. J. Math. **90** (1968), 511–521.

[Ha1] R. Hartshorne, *Connectedness of the Hilbert scheme*, Publ. Math. de I.H.E.S. **29** (1966), 261–304.

[Ha2] ______, *Algebraic geometry*, G.T.M. 52, Springer, 1977.

[Ia1] A. Iarrobino, *Reducibility of the families of* 0*–dimensional schemes on a variety*, Inventiones Math. **15** (1972), 72–77.

[Ia2] ______, *Punctual Hilbert schemes*, Mem. Amer. Math. Soc., vol. 10, A.M.S., 1977.

[Ia3] ______, *Compressed algebras: Artin algebras having given socle degrees and maximal length*, Trans. Amer. Math. Soc., vol. 285, A.M.S., 1984.

[Ia4] ______, *Associated graded algebra of a Gorenstein Artin algebra*, Mem. Amer. Math. Soc., vol. 107, A.M.S., 1994.

[I–E] A. Iarrobino, J. Emsalem, *Some zero–dimensional generic singularities; finite algebras having small tangent space*, Compos. Math. **36** (1978), 145–188.

[I–K] A. Iarrobino, V. Kanev, *Power sums, Gorenstein algebras, and determinantal loci.*, L.M.N., vol. 1721, Springer, 1999.

[HK] H. Kleppe, *Deformations of schemes defined by vanishing of pfaffians*, J. Algebra **53** (1978), 84–92.

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